Optimal Line Breaking for Music

Here's a formulation of the problem of typesetting bars of music on lines of constant width[1]. We are given n bars $\langle b_1, b_2, ..., b_n \rangle$ and their lengths $\langle l_1, l_2, ..., l_n \rangle$. The width of the page is W.

A *solution* to the problem partitions the bars into lines, and is represented by $\langle a_1, a_2, ..., a_k \rangle$, where $1 = a_1 < a_2 < \cdots < a_k = n + 1$. The bars on a line must fit, i.e., $l_{a_h} + l_{a_h+1} + \cdots + l_{a_{h+1}-1} \leq W$, for $1 \leq h < k$. The *penalty* for each line reflects how badly the bars on it fit, and is given by $\Phi(W - (l_{a_h} + l_{a_h+1} + \cdots + l_{a_{h+1}-1}))$, where Φ is a monotonically non-decreasing function such as $\Phi(x) = x^2$.

Our objective is to find a solution that minimizes the total penalty:

$$\sum_{h=1}^{k-1} \Phi(W - (l_{a_h} + l_{a_h+1} + \cdots + l_{a_{h+1}-1}))$$

Here's a dynamic programming solution to this problem[2]. Let P_i be the minimum penalty for the subproblem $\langle b_i, b_{i+1}, \ldots, b_n \rangle$. Given i, let q, $i \leq q \leq n$, be the largest index such that $l_i + l_{i+1} + \ldots l_q \leq W$. Then,

$$P_i = \min_{i \leq j \leq q} \{\Phi(W - (l_i + l_{i+1} + \ldots l_j)) + P_{j+1}\}.$$

I.e., to solve the subproblem $\langle b_i, b_{i+1}, \ldots, b_n \rangle$, we place a number of bars starting from i on the same line; then we solve the subproblem consisting of the remaining bars recursively. If the last line need not span the entire width of the page, let $P_r = P_{r+1} = \cdots P_n = 0$, where r is the smallest index such that $l_r + l_{r+1} + \cdots + l_n \leq W$. Otherwise just let $P_{n+1} = 0$. The minimal penalty will be given by P_1. The algorithm runs in $O(nm)$ time, where m is the maximum number of bars that will fit on a line.

[2]Adapted from http : //www.cs.rochester.edu/u/ogihara/cs282/slides/C10.ps.